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Counting Symmetry-Breaking Solutions to Symmetric Variational Problems

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By combining Michel's geometric theory of symmetry breaking and classical results from variational analysis, we obtain a lower bound on the number of critical points with given symmetry $H \subseteq G$ of a potential symmetric under G. The result is obtained by applying the Ljusternik–Schnirelman category in the group orbit space, and can be extended along the same lines to more general situations.

1. INTRODUCTION

It is well known that Michel's theorem (1971a) [see also Michel (1971b) and Abud and Sartori (1983) for more detailed discussion, and Michel (1980), Michel and Radicati (1971a,b, 1973), Cabibbo and Maiani (1971), and Sartori (1992) for applications in theoretical physics] permits one to obtain generic minimal symmetry-breaking solutions to symmetric variational problems on the basis of a study of the geometry of group action, and in particular of the *isotropy stratification* (Michel, 1971a,b; Abud and Sartori, 1983) of the orbit space.

Here, by "minimal symmetry breaking" we mean a breaking of the symmetry from G to G_0 with G_0 a maximal isotropy subgroup of G; in the Michel theory language this would be more precisely expressed by saying that G_0 corresponds to the symmetry type of a maximally singular stratum. By "generic" solution, we mean a solution which depends only on the symmetry properties of the problem, but not on the particular form of the potential.

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Thus, Michel's theory provides *model-independent* solutions to variational problems.[3] The best-known application of Michel's theorem, and the one which motivated Michel's work, is maybe the one by Michel and Radicati (1971a,b, 1973; Michel, 1980) to the spontaneous symmetry breaking in the $SU(3)$ theory of elementary particles, giving raise to the octet.[4]

The purpose of the present paper is to remark that, by the same approach, one can also count the model-independent number of (not necessarily minimal) symmetry-breaking solutions. For a variational problem, critical points come in group orbits, so that we will actually count critical orbits with a given symmetry type, as explained below.

We will state our result in the case of a finite-dimensional smooth manifold M on which is defined the smooth action of a compact Lie group G (one says then that M is a smooth G-manifold). However, our results would extend to a much more general situation (e.g., Banach manifolds), provided some technical conditions are met; this extension would essentially parallel the extension of Michel's theorem to the "symmetric criticality principle" of Palais (1979, 1984),[5] and will not be discussed here.

It is remarkable that our result can be stated, in mathematical language, in terms of the Ljusternik–Schnirelman (LS) category (see, e.g., Ambrosetti, 1992), which, to conform to physicists' language, we will call the *LS index*. Actually, progress in this direction has been made in the mathematical literature, but there the focus is on the equivariant LS index in the manifold M rather than, as here, the ordinary LS index in the orbit space $\Omega = M/G$ (see, e.g., Benci, 1981; Benci and Pacella, 1985; Bartsch, 1993; Matzeu and Vignoli, 1994).

In the following, we will first briefly recall Michel's construction of isotropy stratification, and the key lemma used by Michel in order to prove his theorem (Michel, 1971a). Our present result will then follow from Michel's discussion, essentially by just applying it (and the LS category) to the problem at hand here.

The Michel construction is discussed in more detail in Michel (1971a), Abud and Sartori (1983), Sartori (1992), and Gaeta (1990, 1992a, 1993), to which we refer for further precision. This same, or some strongly related, construction, and to some extent the same lemma, was then used to obtain further results in the field of nonlinear dynamics, such as the equivariant branching lemma and the reduction lemma, some of which have become

[3] Actually, Michel's theory is able to deal with more general classes of problems with symmetry; however, here we are mainly interested in the variational case.

[4] See also the contemporaneous results of Cabibbo and Maiani (1971).

[5] It is interesting to notice that Cabibbo and Maiani (1971) had used an argument essentially amounting to the symmetric criticality principle, although they did not go to the generality of the Michel theorem. The strange story of the argument will be discussed elsewhere.

fundamental tools in understanding bifurcation theory and symmetry breaking (see, e.g., Gaeta, 1990, 1992b; Cicogna, 1981, 1990; Vanderbauwhede, 1982; Golubitsky and Stewart, 1985; Stewart, 1988; Golubitsky *et al.*, 1988). One should also mention the deep results of Field and Richardson (1989, 1990, 1992a,b), who extended Michel's theory and clarified the role of the Weyl group in this context.

We make no use of the Morse theory (Matzeu and Vignoli, 1994; Morse and Cairns, 1969; Milnor, 1969), so that we cannot reproduce results that intrinsically need it (but one could apply the Morse theory to the present construction, i.e., in Ω). Also, we do not discuss here periodic solutions, which have been recently studied by paralleling Michel's approach (Koenig and Chossat, 1994; Chossat *et al.*, 1995; see also Melbourne, 1994) or can be studied by the S^1-equivariant index (Benci, 1981; Benci and Pacella, 1985; Bartsch, 1993; Matzeu and Vignoli, 1994).

2. ISOTROPY STRATIFICATION AND ORBIT SPACE

Let M be a finite-dimensional smooth manifold and G a compact Lie group; let a smooth action (not necessarily linear) of G on M be defined, i.e., let M be a G-manifold. To any point $x \in M$ is then associated a subgroup $G_x \subseteq G$, its *isotropy subgroup*:

$$G_x := \{g \in G: gx = x\} \tag{1}$$

In many cases, and anyway in the present context, subgroups conjugate in G should be seen as physically equivalent,[6] so that we will consider equivalence classes of subgroups under conjugation in G, called *symmetry types*:

$$[H] := \{K \subseteq G: K = gHg^{-1}, g \in G\} \tag{2}$$

If there exist $G_1 \in [H]$, $G_2 \in [K]$ such that $G_1 \subset G_2$, we say that the symmetry type $[K]$ is higher than the symmetry type $[H]$, or $[H] < [K]$.

We can then consider an equivalence relation in M as follows: the equivalence class of x, called the *stratum*[7] of x, denoted by $\sigma[x]$, is defined as

$$\sigma[x] := \{y \in M: G_y \in [G_x]\} \equiv \{y \in M: G_y = gG_xg^{-1}, g \in G\} \tag{3}$$

The set $\sigma[x]$ is a smooth submanifold of M (Michel, 1971b; Abud and Sartori, 1983; Bredon, 1972; Palais and Terng, 1988).

Let us also consider another equivalence relation on M given by the G-action: the equivalence classes are the G-orbits, denoted by $\omega[x]$,

$$\omega[x] := \{Gx\} \equiv \{y \in M: y = gx, g \in G\} \tag{4}$$

[6] E.g., for $G = SO(3)$, the subgroups $H_l = SO(2)$ of rotation around the axis l.
[7] This is the latin word for layer (plural: *strata*).

It is quite obvious, but important, that

$$\omega[x] \subseteq \sigma[x] \tag{5}$$

(this simply follows from the fact that $y = gx$ implies $G_y = gG_x g^{-1}$).

One can also consider the *orbit space*

$$\Omega := M/G \tag{6}$$

[which with the present assumptions on M and G is a semialgebraic manifold (Abud and Sartori, 1983; Sartori, 1992; Bredon, 1972; Palais and Terng, 1988; Hilbert, 1897], and define a stratification on this: that this is possible is indeed granted by (5). We will denote isotropy strata in Ω by Σ, and have

$$\Sigma[\omega] := \{\nu \in \Omega : \exists x \in \omega, \exists y \in \nu : y \in \sigma[x]\} \tag{7}$$

Again, $\Sigma[\omega]$ is a smooth submanifold of Ω.

An important point is that the group–subgroup relation is reflected into a bordering relation for strata. This means that if G_x, G_y are isotropy subgroups of G and ω, $\nu \in \Omega$ with $x \in \omega$, $y \in \nu$, then $\Sigma[\omega] \subseteq \partial(\Sigma[\nu])$. Conversely, $\partial(\Sigma[\nu])$ belongs to the union of strata $\Sigma[\omega_i]$ with symmetry types $[K] \geq [G_y]$.

3. EQUIVARIANT FLOWS AND INVARIANCE OF STRATA

The relevance of the above construction and definitions in this context is due to the following result. Consider a vector field f: $M \to TM$ on M, or equivalently the dynamical system on M defined by f,

$$\dot{x} = f(x) \tag{8}$$

and suppose that f is G-equivariant, i.e.,

$$f(gx) = (\mathrm{d}g \cdot f)(x) \tag{9}$$

where dg denotes the action of g on the tangent space (recall that the action of G is not necessarily linear). Then (Michel, 1971a) necessarily

$$f: \quad x \to \mathrm{T}_x\sigma[x] \tag{10}$$

In the case of a variational G-invariant problem defined by a smooth potential F: $M \to R$, i.e., a potential which satisfies

$$F(gx) = F(x) \qquad \forall g \in G, \quad \forall x \in M \tag{11}$$

the gradient $f := \nabla F$ of F satisfies (9), and therefore (10). This means that, given a stratum $\sigma \subseteq M$, we can consider the restriction F_σ of F to σ: critical points of F_σ are then granted to be critical points[8] of F as well (Michel, 1971a; Palais, 1979, 1984).

Notice that (11) also grants the critical points of F come in G-orbits, and that we can think of F as being defined on Ω rather than on M; we use then the notation $\Phi(\omega)$, with Φ: $\Omega \to R$ and

$$\Phi(\omega) = F(x), \qquad x \in \omega \tag{12}$$

Consider now the vector field ϕ induced by f on Ω; if $f = \nabla F$, then $\phi = \nabla_\omega \Phi$, where ∇_ω is the gradient with respect to ω.[9] As critical points of F come in G-orbits, we can equivalently study critical points of Φ.

Applying (10) to $f = \nabla F$, we have that

$$(\nabla_\omega \Phi)(\omega) \in T_\omega \Sigma[\omega] \tag{13}$$

Therefore, we can consider the restriction Φ_Σ of Φ to any stratum $\Sigma \subseteq \Omega$; critical points of Φ_Σ will also be critical points of Φ, corresponding to critical points of F with symmetry type

$$[H] = [G_x], \qquad x \in \omega, \quad \omega \in \Sigma \tag{14}$$

It should be stressed that, in general, the strata σ or Σ are *not* closed manifolds; thus, although (10) ensures that the solution $x(t)$ of (8) with initial datum $x(0) = x_0$ lies in $\sigma_0 = \sigma[x_0]$, even if $x(t)$ admits a limit x_* for $t \to \infty$, this could very well not lie in σ_0, but only in $\bar{\sigma}_0$, i.e. it could be $x_* \in \partial\sigma_0$, $x_* \notin \sigma_0$.

Similar considerations also hold for the flow ϕ, and in particular the gradient flow $\phi = \nabla_\omega \Phi$, in Ω considered in (13) and in the following.

Finally, it should be mentioned that (10) could be made more precise (Golubitsky and Stewart, 1985; Stewart, 1988; Golubitsky *et al.*, 1988); indeed, if M_H is the subset of M left fixed by $H \subseteq G$, i.e.,

$$M_H := \{x \in M\colon H \subseteq G_x\} \tag{15}$$

and writing M_x for M_{G_x}, we have that if f satisfies (9), then[10]

$$f\colon \quad x \in T_x M_x \subseteq T_x \sigma[x] \tag{16}$$

Notice, however, that in this way we lose track of the physically desirable identification of conjugate subgroups.

[8] It should be stressed that this does not extend to stability matters: e.g., a minimum of F_σ will not necessarily be a minimum of F: it could also be a saddle point.

[9] This could be precisely defined, e.g., by using a Hilbert minimal integrity basis (Michel, 1971b; Abud and Sartori, 1983; Sartori, 1992; Bredon, 1972; Palais and Terng, 1988; Hilbert, 1897).

[10] This fact is related to the slice theorem; see, e.g., Abud and Sartori (1983), Bredon (1972), and Palais and Terng (1988).

4. COUNTING CRITICAL ORBITS, AND SYMMETRY-BREAKING SOLUTIONS

Let us now introduce the following notation: we denote by Σ^H the stratum in Ω whose orbits have symmetry type $[H]$,

$$\Sigma^H := \Sigma[\omega], \qquad \omega: \quad \forall x \in \omega, H \in [G_x] \tag{17}$$

and we denote Φ_{Σ^H} simply by Φ_H for ease of notation.

The above discussion, which followed Michel's construction (Michel, 1971a,b; Abud and Sartori, 1983; Sartori, 1992; Gaeta, 1990, 1992a, 1993), shows in particular the following result.

Lemma 1. The number $n[H]$ of critical G-orbits of F in M with symmetry type $[H]$ is equal to the number of critical points of Φ_H.

Indeed, we have seen that critical G-orbits of F are critical points of Φ; also, by definition, critical orbits of F with symmetry type $[H]$ are critical points of Φ which lie in Σ^H. But, as we have seen, these are also critical points of Φ_H: hence the lemma is just a reformulation of the previous discussion.

Due to the considerations appearing after equation (14), we are also interested in considering the set Λ^H corresponding to the union of the Σ^K with symmetry type $[H]$ or higher,

$$\Lambda^H := \bigcup_{[H]\le[K]} \Sigma^K \tag{18}$$

Notice that we can easily define the restriction of Φ to Λ^H, being given in terms of the Φ_H with $[H] \le [K]$; this will be called Ψ_H.

From these definitions we have immediately, as a corollary of Lemma 1 and with the same proof, the following result.

Lemma 2. The number of $k[H]$ of critical G-orbits of F in M with symmetry type $[H]$ or higher is equal to the number of critical points of Ψ_H.

We want to consider the situation in which either M is closed or there is a closed submanifold $M_0 \subseteq M$ which is invariant under both the G-action and the gradient flow $f = \nabla F$, i.e., such that $G(M_0) = M_0$ and on ∂M_0 the gradient of F in the outward normal direction has definite sign. For ease of notation, we just consider the case M is closed (the other case being exactly the same provided we restrict our considerations to M_0), and that if ∂M exists, $f = -\nabla F$ points inward on ∂M; we say then that M is contracting under F.

In this case, we are guaranteed that there is at least a critical point of F in the interior of M_0, and topological considerations can also guarantee there is a higher number of critical points on M.

It should be stressed that now the difference between Lemmas 1 and 2 is that the first applies to a potential Φ_H which could well not have critical points in its domain of definition Σ^H, while in the second case we are guaranteed Ψ_H has critical points in its domain of definition Λ^H.

The critical points x_i of F will have a symmetry type $[G_{x_i}]$, and we are interested precisely in this, i.e., we are interested in the *symmetry breakings* corresponding to critical points of the symmetric potential F.

Michel's theorem (Michel, 1971a) allows us to conclude that if an orbit ω_0 is isolated in its stratum, then ω_0 is a critical point of any Φ, and correspondingly it is a critical orbit of any G-invariant potential F. Here we do not aim at such a precise identification of critical orbits, but rather at identifying only the symmetry types of critical orbits, and the number $n[H]$ of critical orbits with symmetry type $[H]$; we are also interested in the number $\nu[H]$ of critical orbits with symmetry type $[H]$ or higher, $\nu[H] = \Sigma_{[H]\leq[K]} n[H]$.

We would like in particular to know what is the minimal possible value of $n[H]$, i.e., the number of critical G-orbits for F with given symmetry type $[H]$ which is possible independent[11] of the actual F, just on the basis of the symmetry under G; we denote this minimal value of $n[H]$ by $m[H]$. Similarly, we would also like to know the minimal possible value of $\nu[H]$, which we call $\mu[H]$.

Let us first decompose Λ^H into disjoint connected components $\Lambda_a^H \subseteq \Omega$, where $a = 1, \ldots, c(H)$; we can then consider separately the Λ_a^H.

We recall that the LS category (or LS index) (Ambrosetti, 1992; Matzeu and Vignoli, 1994) of a connected set $A \subseteq X$, denoted by $\mathcal{LS}(A, X)$, is defined as the minimal number k such that there exists a covering of A by closed sets A_i,

$$A = A_1 \cup \cdots \cup A_k \tag{19}$$

with all the A_i's contractible in X.

A classical result in variational analysis (Ambrosetti, 1992), and more specifically in LS theory, is that *a scalar function p: $A \to R$ has at least $\mathcal{LS}(A, X)$ critical points in A.*

Thus, applying this to Λ_a^H and Φ_H, we have that if Λ_a^H is contractible, we have at least a critical point of Φ_H in it, and if Λ_a^H is noncontractible, we have at least as many critical points of Φ_H as we need contractible open sets to cover Λ_a^H. Therefore, the above result from LS theory permits us to conclude immediately that:

Lemma 3. The number $\mu[H]$ defined above is given by

$$\mu[H] = \sum_{a=1}^{c(H)} \mathcal{LS}(\Lambda_a^H, \Omega) \tag{20}$$

[11] In this context, the original formulation of Michel's theorem (Michel, 1971a) guarantees that if Σ^H is the union of h isolated points $(\omega_1, \ldots, \omega_h)$, as each of these is critical, then $h[H] = h$. The result we obtain below is a direct generalization of this.

We can therefore consider the lattice of isotropy subgroups of G and use the above result to study symmetry breakings under F. For the sake of definiteness, we will consider the minima of F, and assume that if M has a border, then $f = \nabla F$ points outward from M on ∂M, i.e., M is contracting under F.

Let us first consider G itself, and correspondingly $\Lambda^G = \Sigma^G \subseteq \Omega$ if it exists. In this case, we are sure there is at least a critical point $\omega \in \Lambda^G$ (indeed Σ^G is a maximally singular stratum, if it exists), and actually we know that there are at least $\mu[G] = \mathcal{LS}(\Sigma^H, \Omega)$ critical points of Φ with symmetry type $[G]$ (i.e., invariant under G, as $H \in [G] \Leftrightarrow H = G$ since G is the full group).

Assume now that the critical points in Λ^G are known to be maxima, i.e., that the symmetry G is broken. Let us consider a maximal isotropy subgroup H of G. We can now consider Λ^H and claim that $\mu[H] = \mathcal{LS}(\Lambda^H, \Omega)$, but the information that the symmetry G is broken allows us also to consider $\Sigma^H = \Lambda^H \backslash \Lambda^G$ being guaranteed that Σ^H is contracting under Φ. Thus, we can also affirm that there are at least $m[H] = \mathcal{LS}(\Sigma^H, \Omega)$ critical points of Φ with symmetry type H.

It is clear that the procedure can be iterated along any chain of subgroups. In this way, we arrive at the following conclusion:

Lemma 4. Let $H = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_n \subseteq G$ be a complete chain of isotropy subgroups, i.e., all the H_i are isotropy subgroups of G for the G-action on M, and H_i is a maximal isotropy subgroup of H_{i+1} (H_n is a maximal isotropy subgroup of G if there is no $x \in M$ for which $G_x = G$). Assume that the symmetries $H_1, \ldots, H_n$ are broken, i.e., that the critical points of Φ with symmetry types $[H_1], \ldots, [H_n]$ are known to be unstable. Then, there are at least

$$m[H] = \mathcal{LS}(\Sigma^H, \Omega) \tag{21}$$

critical points of Φ with symmetry type $[H]$.

Example. As a simple example, let us consider R^2, on which we take coordinates (x, y); and $G = Z_2 \times Z_2 = Z_2^{(x)} \times Z_2^{(y)}$, with $Z_2^{(x)}$ generated by h: $(x, y) \to (-x, y)$ and $Z_2^{(y)}$ generated by k: $(x, y) \to (x, -y)$. As for the orbit space Ω, this can be identified with the first quadrant of R^2,

$$\Omega \simeq R_{++} = \{(x, y): x \geq 0, y \geq 0\}$$

We have the following lattice of isotropy subgroups:

$$\{e\} \subset \begin{Bmatrix} Z_2^{(x)} \\ Z_2^{(y)} \end{Bmatrix} \subset G$$

It is immediate to check that there are four strata in M [we denote by $\xi = (x, y)$ points of R^2]:

$$\begin{array}{lll} \sigma_0 = \{(0, 0)\} & & G_\xi = G \\ \sigma_1 = \{(0, y)\} & (y \neq 0) & G_\xi = Z_2^{(x)} \\ \sigma_2 = \{(x, 0)\} & (x \neq 0) & G_\xi = Z_2^{(y)} \\ \sigma_3 = \{(x, y)\} & (xy \neq 0) & G_\xi = \{e\} \end{array}$$

The strata Σ_i in Ω are immediately obtained from these by considering $\sigma_i \cap R_{++}$. Notice that $Z_2^{(x)}$ and $Z_2^{(y)}$, although isomorphic, are *not* conjugated in G, so that we have different strata corresponding to these.

The G-invariant potentials $V(x, y)$ are of the form $V = V(x^2, y^2)$, and the requirement that there is an invariant $M \subseteq R^2$ is satisfied if, e.g., $\lim_{x,y\to\infty} V = \infty$, i.e., if V is convex at infinity, which we assume to be the case; equivalently, we choose a rectangle $M = I_x \times I_y$ such that ∇V points outward on ∂M.

Clearly, any $V = V(x^2, y^2)$ has a critical point at the origin. It is also clear that on the x axis and on the y axis, the gradient ∇V is directed along the axis itself, so that if the origin is a maximum, there has to be (at least) a minimum on the x axis for positive y and one for negative y, which are related by k; and similarly on the y axis we have (at least) a minimum for positive x and one for negative x, related by h.

Indeed, applying Lemma 4, we get exactly this result: considering any compact $M \subset R^2$, the LS indexes of Σ_i are all equal to one; then, if the origin is an unstable critical point, we have one critical orbit (at least) in each of σ_1 and σ_2; and if these are also unstable, we get one critical orbit (at least) in the generic stratum σ_4.

5. DISCUSSION AND GENERALIZATIONS

Notice that Lemma 4 has a "global" formulation, in the sense that we require *all* the critical points with higher symmetry $[H_1], \ldots, [H_n]$ to be unstable. However, it is quite clear that we could equally well make "local" equivalent considerations, i.e., restrict our attention to a neighborhood U_0 of a given critical point ω_0 of symmetry type $[G_0]$ when ω_0 is unstable, or to a neighborhood $U_1 \subseteq U_0$ of a given critical point $\omega_1 \in U_0$ if both ω_0 and ω_1 are unstable, and so forth. (Here each G_j plays the role of the H_{n-j} in the previous formulation).

One could also consider the case that F—and therefore Φ—depends on a control parameter λ in such a way that for varying λ the stability of critical points is changed. We could use the approach sketched above to follow a chain of symmetry breakings, or in mathematical language, a chain of bifurcations; such an extension would just repeat Lemma 4 in a "local" formulation, requiring, moreover, the existence of a suitable chain of λ-dependent neighborhoods.

In this way, one could count—in terms of the LS index—the generic number of bifurcating branches with a given symmetry type at the kth symmetry breaking in the chain. This would be a generalization of the "equivariant branching lemma" (EBL) of Cicogna (1981, 1990) and Vanderbauwhede (1982) in the frame of variational (stationary) bifurcation problems.

It should be mentioned that the EBL was also generalized to study the case of Hopf bifurcation (of periodic solutions) and of quaternionic bifurcation (Golubitsky and Stewart, 1985; Stewart, 1988; Golubitsky *et al.*, 1988; Milnor, 1969; Koenig and Chossat, 1994). It is then natural to ask if the approach developed in the present paper could also be generalized to such a setting. I have no answer to such a question at this stage, although a natural way of attempting such an extension would combine the present approach, the S^1 index of Benci (1981; Benci and Pacella, 1985; Bartsch, 1993), and the "splitting principle" recently proposed in Gaeta (1994).

The S^1 symmetry of the Benci index would be a dynamical one, i.e., correspond to motion along the periodic solutions rather than being a symmetry in the sense considered here. Clearly, the two notions can coincide, i.e., one can have a periodic orbit which lies in one G-orbit; in this case one has a *relative equilibrium*. It appears that combining the S^1 index and the splitting principle would be particularly efficient in considering bifurcation of[12] relative equilibria.

In many cases, one is interested in variational problems in infinite-dimensional spaces, e.g., spaces of a section of a fiber bundle (as in gauge theories). Although the extension of the Michel theory to such a setting is very difficult—and a full extension probably impossible (Gaeta, 1992a, 1993)—as the stratification is not properly defined in this case, the equivalent of (16) does still hold, and yields the Palais *symmetric criticality principle*; also, a partial extension of the Michel theory to this context (essentially, for maximal isotropy subgroups) is possible (Gaeta, 1992a, 1993). Thus, the approach proposed here can also be extended to infinite-dimensional variational problems, essentially in the cases—and with the limitations—considered by Palais (1979, 1984).

As a final remark, already anticipated in the introduction, we notice that if we want to apply the Morse theory, by using the same construction we can apply it to Φ, i.e., in Ω, rather than to F, i.e., in M. Similarly, if we want

[12] This should not be confused with the bifurcation *from* relative equilibria, for which see the paper by Krupa (1990).

specifically to look for periodic solutions (in a nongradient system) we could apply the S^1-index (Benci, 1981; Benci and Pacella, 1985; Bartsch, 1993; Matzeu and Vignoli, 1994) to ϕ, i.e., in Ω, rather than to f, i.e., in M.

In concluding, more than stressing the strict content of the previous lemmata, we would like to emphasize the main idea—and result—of the present approach: namely, that we can use the *ordinary* LS theory (as opposed to the equivariant one) to study symmetric variational problems and symmetry-breaking solutions, provided we work in Ω rather than in M, and we utilize Michel's theory.

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REFERENCES

Abud, M., and Sartori, G. (1983). The geometry of spontaneous symmetry breaking, *Annals of Physics*, **150**, 307.

Ambrosetti, A. (1992). Critical points and nonlinear variational problems, *Société Mathématique de France: Mémoires (Nouvelle serie)*, **49**(2).

Bartsch, T. (1993). *Topological Methods for Variational Problems with Symmetries* (Lecture Notes in Mathematics 1560), Springer, Berlin.

Benci, V. (1981). A geometrical index for the group S^1 and some applications to the study of periodic solutions of ordinary differential equations, *Communications in Pure and Applied Mathematics*, **34**, 393.

Benci, V., and Pacella, F. (1985). Morse theory for symmetric functionals on the sphere and an application to a bifurcation problem; *Nonlinear Analysis*, **9**, 763.

Bredon, G. (1972). *Compact Transformation Groups*, Academic Press, New York.

Cabibbo, N., and Maiani, L. (1971). Weak interactions and the breaking of hadronic symmetry, in *Evolution of Particle Physics*, M. Conversi, ed., Academic Press, London.

Chossat, P., Koenig, M., and Montaldi, J. (1995). Bifurcation générique d'ondes rotatives d'isotropie maximale, *Comptes Rendus de l'Academie des Sciences Paris*, **320-I**, 25.

Cicogna, G. (1981). Symmetry breakdown from bifurcation, *Lettere al Nuovo Cimento*, **31**, 600.

Cicogna, G. (1990). A nonlinear version of the equivariant bifurcation lemma, *Journal of Physics A*, **23**, L1339.

Field, M., and Richardson, R. (1989). Symmetry breaking and the maximal isotropy subgroup conjecture for reflection groups, *Archive for Rational Mechanics and Analysis*, **105**, 61.

Field, M., and Richardson, R. (1990). Symmetry breaking in equivariant bifurcation problems, *Bulletin of the American Mathematical Society*, **22**, 79.

Field, M., and Richardson, R. (1992a). Symmetry breaking and branching patterns in equivariant bifurcation theory I, *Archive for Rational Mechanics and Analysis*, **118**, 297.
Field, M., and Richardson, R. (1992b). Symmetry breaking and branching patterns in equivariant bifurcation theory II, *Archive for Rational Mechanics and Analysis*, **120**, 147.
Gaeta, G. (1990). Bifurcation and symmetry breaking, *Physics Reports*, **189**, 1.
Gaeta, G. (1992a). Michel theorem and critical orbits for gauge functionals, *Helvetica Physica Acta*, **65**, 922.
Gaeta, G. (1992b). Reduction and equivariant branching lemma: Dynamical systems, evolution equations, and gauge theories, *Acta Applicandae Mathematicae*, **28**, 43.
Gaeta, G. (1993). Critical sections of gauge functionals: A symmetry approach, *Letters in Mathematical Physics*, **28**.
Gaeta, G. (1995). A splitting lemma for equivariant dynamics, *Letters in Mathematical Physics*, **33**, 313.
Golubitsky, M., and Stewart, I. (1985). Hopf bifurcation in the presence of symmetry, *Archive for Rational Mechanics and Analysis*, **87**, 107.
Golubitsky, M., Schaeffer, D., and Stewart, I. (1988). *Singularities and Groups in Bifurcation Theory*, Vol. II, Springer, New York.
Hilbert, D. (1897). *Theory of Algebraic Invariants*, Gottingen [English translation, Cambridge University Press, Cambridge (1993)].
Koenig, M., and Chossat, P. (1994). Characterisation des bifurcations pour les champs de vecteurs équivariants sous l'action d'un groupe de Lie compact, *Comptes Rendus de l'Adademie des Sciences Paris*, **318-I**, 31.
Krupa, M. (1990). Bifurcations of relative equilibria, *SIAM Journal on Mathematical Analysis*, **21**, 1453.
Matzeu, M., and Vignoli, A., eds. (1994). *Topological Nonlinear Analysis: Degree, Singularity, and Variations*, Birkhauser, Basel.
Melbourne, I. (1994). Maximal isotropy subgroups for absolutely irreducible representations of compact Lie groups, *Nonlinearity*, **7**, 1385.
Michel, L. (1971a). Points critiques des fonctions invariants sur une *G*-variété, *Comptes Rendus de l'Academie des Sciences Paris A*, **272**, 433.
Michel, L. (1971b). Nonlinear group action. Smooth action of compact Lie groups on manifolds, in *Statistical Mechanics and Field Theory*, R. N. Sen and C. Weil, eds., Jerusalem.
Michel, L. (1980). Symmetry defects and broken symmetry. Configurations. Hidden Symmetry, *Review of Modern Physics*, **52**, 617.
Michel, L., and Radicati, L. (1971a). Properties of the breaking of hadronic internal symmetry, *Annals of Physics*, **66**, 758.
Michel, L., and Radicati, L. (1971b). Breaking of the $SU(3) \times SU(3)$ symmetry in hadronic physics, in *Evolution of Particle Physics*, M. Conversi, ed., Academic Press, London.
Michel, L., and Radicati, L. (1973). The geometry of the octet, *Annales de l'Institut Henri Poincaré*, **18**, 185.
Milnor, J. W. (1969). *Morse Theory*, Princeton University Press, Princeton, New Jersey.
Morse, M., and Cairns, S. S. (1969). *Critical Point Theory in Global Analysis and Differential Topology*, Academic Press, London.
Palais, R. S. (1979). The principle of symmetric criticality, *Communications in Mathematical Physics*, **69**, 19.
Palais, R. S. (1984). Applications of the symmetric criticality principle in mathematical physics and differential geometry, in *Proceedings of the 1981 Shangai Symposium on Differential Geometry and Differential Equations*, Gu Chaohao, ed., Science Press, Beijing.
Palais, R., and Terng, C. L. (1988). *Critical Point Theory and Submanifold Geometry* (Lecture Notes in Mathematics 1353), Springer, Berlin.

Sartori, G. (1992). Geometric invariant theory: A model-independent approach to symmetry and/or supersymmetry breaking, *Nuovo Cimento* **14**(11).

Schwarz, G. (1975). Smooth functions invariant under a compact Lie group, *Topology,* **14**, 63.

Schwarz, G. (1980). Lifting smooth homotopies of orbit spaces, *Publications Mathématiques I.H.E.S.*, **51**, 37.

Stewart, I. (1988). Bifurcation with symmetry, in *New Directions in Dynamical Systems,* T. Bedford and J. Swift, eds., Cambridge University Press, Cambridge.

Vanderbauwhede, A. (1982). *Local Bifurcation and Symmetry,* Pitman, Boston.